From Boltzmann to Navier-Stokes and Euler

1. The distribution function and the Boltzmann equation

Define the distribution function $f(\vec{x}, \vec{v}, t)$ such that $f(\vec{x}, \vec{v}, t)d^3xd^3v$ = probability of finding a particle in phase space volume d^3xd^3v centered on $\vec{x}, \vec{v}$ at time t.
The normalization is

$$\int\int f(\vec{x}, \vec{v}, t)d^3xd^3v = N \text{ (\# particles in system)}$$

and

$$[f] = \text{ cm}^{-3}(\text{ cm s}^{-1})^{-3}.$$

Particles are not created or destroyed, so continuity implies

$$\frac{\partial f}{\partial t} + \sum_{i=1,3}\left(\dot{x}_i \frac{\partial f}{\partial x_i} + \dot{v}_i \frac{\partial f}{\partial v_i}\right) = \frac{\mathrm{d}f}{\mathrm{d}t}|_c, \tag{1}$$

where $\frac{\mathrm{d}f}{\mathrm{d}t}|_c$ represents discontinuous motion of particles through phase space because of collisions. (Collisions cannot instantaneously change particle positions, but they can instantaneously change particle velocities.)
Substituting $\dot{x}_i = v_i$ and $\dot{v}_i = g_i$ leads to the Boltzmann equation

$$\frac{\partial f}{\partial t} + \sum_i v_i \frac{\partial f}{\partial x_i} + \sum_i g_i \frac{\partial f}{\partial v_i} = \frac{\mathrm{d}f}{\mathrm{d}t}|_c. \tag{2}$$

Stellar dynamics is based on the collisionless Boltzmann equation, with the RHS=0.
In the fluid ($\lambda \ll L$) limit, on the other hand, the collision term makes $f(\vec{v})$ approximately Maxwellian while locally conserving mass, momentum, and energy.
The Boltzmann equation is hard to manage because it is 6-d, but it tells us more than we really want to know in most cases.
We are usually happy with the density, mean velocity, and velocity dispersion as a function of $\vec{x}$, since the velocity distribution function is close to Maxwellian.
We can therefore get more useful equations by taking moments of the Boltzmann equation.
A similar procedure is used in stellar dynamics to obtain the Jeans equations. However, these are less powerful than the hydrodynamic equations because in the absence of collisions the velocity distribution function may be far from Maxwellian (in particular, it may be anisotropic).

The mass density is

$$\rho(\vec{x}, t) = \int mf(\vec{x}, \vec{v}, t)d^3v, \tag{3}$$

where m is the particle mass. If necessary, one can sum over different f's for different particle types, but we will assume a single particle species here.
The mass-weighted average of a quantity Q at position $\vec{x}$ is

$$\langle Q \rangle = \frac{1}{\rho} \int Q m f(\vec{x}, \vec{v}, t) d3v. \tag{4}$$

2. The continuity equation

Multiply both sides of the Boltzmann equation by m and integrate over d^3v:

$$\frac{\partial}{\partial t} \int m f d^3v + \sum_i \frac{\partial}{\partial x_i} \int m f v_i d^3v + m \int \sum_i \frac{\partial}{\partial v_i} (g_i f) d^3v = \int m \frac{\mathrm{d}f}{\mathrm{d}t}|_c d^3v.$$

First term is

$$\frac{\partial \rho}{\partial t}.$$

Second term is

$$\sum_i \frac{\partial}{\partial x_i} \left(\rho \langle v_i \rangle \right) \equiv \vec{\nabla} \cdot \left(\rho \vec{u} \right),$$

where

$$\begin{aligned} \vec{u} &\equiv \langle \vec{v} \rangle = \text{mean fluid velocity at } (\vec{x}, t)\ [\mathrm{cm\,s^{-1}}] \\ \vec{w} &\equiv \vec{v} - \vec{u} = \text{particle random velocity } [\mathrm{cm\,s^{-1}}]. \end{aligned}$$

Third term is

$$m \int_V \vec{\nabla}_v \cdot (\vec{g} f) d^3v = m \int_S \hat{\mathbf{n}} \cdot (\vec{g} f) dA = 0.$$

The first equality follows from the divergence theorem and the second from the assumption that f vanishes as $v \longrightarrow \infty$.

RHS vanishes because of local mass conservation: collisions do not create or destroy particles at a fixed position, only shift them in velocity space.

Result:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0.$$

More transparently

$$\frac{\partial \rho}{\partial t} + \vec{u} \cdot \vec{\nabla} \rho + \rho \vec{\nabla} \cdot \vec{u} = 0,$$

implying a continuous change of mass density.

3. The momentum equation

Multiply Boltzmann equation by $m\vec{v}$ and integrate.

$$\frac{\partial}{\partial t}\int mv_j f d^3v + \sum_i \frac{\partial}{\partial x_i}\int mfv_jv_i d^3v + m\int \sum_i g_i v_j \frac{\partial f}{\partial v_i} d^3v = \int mv_j \frac{\mathrm{d}f}{\mathrm{d}t}|_c d^3v.$$

First term is

$$\frac{\partial}{\partial t}(\rho u_j).$$

Second term is

$$\sum_i \frac{\partial}{\partial x_i}\left(\rho\langle v_j v_i\rangle\right) = \sum_i \frac{\partial}{\partial x_i}\left(\rho u_i u_j + \rho\langle w_i w_j\rangle\right).$$

For the third term, note that

$$\frac{\partial}{\partial v_i}\left(v_j f\right) = v_j \frac{\partial f}{\partial v_i} + \delta_{ij} f$$

to obtain

$$\sum_i mg_i \int \left[\frac{\partial}{\partial v_i}\left(v_j f\right) - \delta_{ij} f\right] d^3v = -\sum_i g_i \delta_{ij}\int mf d^3v = -\rho g_j,$$

where $\delta_{ij} \equiv 1$ for $i = j$, 0 for $i \neq j$, and we have used the divergence theorem to get rid of the first term in the brackets.

RHS vanishes because collisions conserve momentum. (Note: Ryden sets the collision term to zero before integrating on the grounds that it will be zero in equilibrium, but I think this is not justified in all circumstances.)

Result:

$$\frac{\partial}{\partial t}\left(\rho u_j\right) + \sum_i \frac{\partial}{\partial x_i}\left(\rho u_i u_j + \rho\langle w_i w_j\rangle\right) = \rho g_j.$$

The diagonal terms of $\langle w_i w_j\rangle$ are generally much larger than the off-diagonal terms, since random velocities in different directions are usually almost uncorrelated.
It therefore makes sense to divide the $\rho\langle w_i w_j\rangle$ term into a contribution from pressure and a contribution from viscosity:

$$P \equiv \quad \text{pressure} \quad = \frac{1}{3}\rho\langle|\vec{w}|^2\rangle \quad [\mathrm{dyne\,cm^{-2}} \text{ or } \mathrm{erg\,cm^{-3}}] \tag{5}$$

$$\pi_{ij} \equiv \quad \text{viscous stress tensor} \quad = P\delta_{ij} - \rho\langle w_i w_j\rangle \quad [\mathrm{dyne\,cm^{-2}} \text{ or } \mathrm{erg\,cm^{-3}}] \tag{6}$$

to obtain

$$\frac{\partial}{\partial t}\left(\rho u_j\right) + \sum_i \frac{\partial}{\partial x_i}\left(\rho u_i u_j + P\delta_{ij} - \pi_{ij}\right) = \rho g_j,$$

or, in tensor form

$$\frac{\partial}{\partial t}(\rho\vec{u}) + \vec{\nabla}\cdot\left(\rho\vec{u}\vec{u} + P\overset{\leftrightarrow}{I} - \overset{\leftrightarrow}{\pi}\right) = \rho\vec{g}.$$

Here $(\vec{u}\vec{u})_{ij} = u_i u_j$, $\overleftrightarrow{I}_{ij} = \delta_{ij}$.
All three tensors are symmetric, and $\overleftrightarrow{\pi}$ is traceless.

One can combine this form of the momentum equation with the continuity equation to get the form

$$\frac{\partial u_j}{\partial t} + \sum_i u_i \frac{\partial u_j}{\partial x_i} = g_j - \frac{1}{\rho} \sum_i \frac{\partial}{\partial x_i} \left(P\delta_{ij} - \pi_{ij} \right)$$

or

$$\frac{\partial \vec{u}}{\partial t} + \left(\vec{u} \cdot \vec{\nabla} \right) \vec{u} = \vec{g} - \frac{1}{\rho} \vec{\nabla} P + \frac{1}{\rho} \vec{\nabla} \cdot \overleftrightarrow{\pi} .$$

Viscosity acts to oppose shearing motion and interpenetration.

4. The energy equation

One can multiply by mv^2 and go through a similar procedure (see Ryden and/or Shu) to derive the internal energy equation

$$\frac{\partial \epsilon}{\partial t} + \vec{u} \cdot \vec{\nabla} \epsilon = -\frac{P}{\rho} \vec{\nabla} \cdot \vec{u} - \frac{1}{\rho} \vec{\nabla} \cdot \vec{F} + \frac{1}{\rho} \Psi$$

$$\epsilon \equiv \quad \text{specific internal energy} \quad = \frac{1}{2} \langle |\vec{w}|^2 \rangle \quad [\mathrm{erg\,g^{-1}}] \tag{7}$$

$$\vec{F} \equiv \quad \text{conduction heat flux} \quad = \frac{1}{2} \rho \langle \vec{w} |\vec{w}|^2 \rangle \quad [\mathrm{erg\,cm^{-3}\,s^{-1}\,cm}] \tag{8}$$

$$\Psi \equiv \quad \text{viscous dissipation rate} \quad = \sum_{i,j} \pi_{ij} \frac{\partial u_i}{\partial x_j} \quad [\mathrm{erg\,cm^{-3}\,s^{-1}}]. \tag{9}$$

One again makes use of the divergence theorem and of the fact that collisions conserve energy (as well as mass and momentum).

Note that if the distribution of $\vec{w}$ is symmetric about zero, then $\vec{F}$ vanishes. If the distribution is skewed, then hot particles have a drift relative to cold particles, producing a heat flux in the direction of the drift.
In most cases, a temperature gradient produces a conductive flux $\vec{F} \propto \vec{\nabla} T$.
However, if $\vec{F}$ is uniform, heat flowing out is replaced by heat flowing in. The local thermal energy changes only if $\vec{\nabla} \cdot \vec{F} \neq 0$.

Ψ represents conversion of bulk motion of the fluid into internal energy via viscous dissipation. It is the viscous analog of heating by PdV work.

5. Eulerian and Lagrangian derivatives

An Eulerian reference frame is one that stays fixed in space. (This has nothing to do with "Euler equations," except that it is the same Euler.)
A Lagrangian reference frame is one that moves with the fluid.

$\frac{\partial Q}{\partial t}$ is the time derivative of Q at a fixed Eulerian position.
The time derivative along an element moving with the mean flow $\vec{u}$,

$$\frac{\mathrm{D}Q}{\mathrm{D}t} = \frac{\partial Q}{\partial t} + \left(\vec{u} \cdot \vec{\nabla}\right) Q, \tag{10}$$

is known as the Lagrangian derivative.

The physical interpretation of hydrodynamics equations may be more transparent in Lagrangian formulation.
Numerical calculations can be performed in either Eulerian or Lagrangian frames. Often one is more powerful than the other for a particular application.
Lagrangian codes tend to be powerful when there is a large dynamic range in density and one needs higher resolution in high density regions than in low density regions.
Smoothed particle hydrodynamics (SPH) is an example of a Lagrangian hydrodynamics algorithm, where one follows fluid properties at the locations of particles, which move with the flow.

6. The Navier-Stokes equations

Collecting our results, we have the Navier-Stokes equations:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{\partial \rho}{\partial t} + \vec{u} \cdot \vec{\nabla}\rho = -\rho\vec{\nabla} \cdot \vec{u}, \tag{11}$$

$$\frac{\mathrm{D}\vec{u}}{\mathrm{D}t} = \frac{\partial \vec{u}}{\partial t} + \left(\vec{u} \cdot \vec{\nabla}\right) \vec{u} = \vec{g} - \frac{1}{\rho}\vec{\nabla}P + \frac{1}{\rho}\vec{\nabla} \cdot \overleftrightarrow{\pi}, \tag{12}$$

$$\frac{\mathrm{D}\epsilon}{\mathrm{D}t} = \frac{\partial \epsilon}{\partial t} + \vec{u} \cdot \vec{\nabla}\epsilon = -\frac{P}{\rho}\vec{\nabla} \cdot \vec{u} - \frac{1}{\rho}\vec{\nabla} \cdot \vec{F} + \frac{1}{\rho}\Psi + \frac{\Gamma - \Lambda}{\rho} \tag{13}$$

$$\Gamma \equiv \text{volumetric radiative heating rate} \quad [\mathrm{erg\,cm^{-3}\,s^{-1}}]$$
$$\Lambda \equiv \text{volumetric radiative cooling rate} \quad [\mathrm{erg\,cm^{-3}\,s^{-1}}].$$

(11) represents local mass conservation. The change in density of a Lagrangian fluid element is produced by a change in the specific volume.

(12) represents local momentum conservation. Accelerations are produced by gravity, pressure gradients, and viscous forces.

(13) represents local energy conservation. Changes in internal energy are produced by PdV work, by conduction, by viscous heating (conversion of bulk kinetic energy), and by radiative heating or cooling, which we have added to our previous results.

$\vec{g}$ may be specified externally or, if self-gravity is important, computed from Poisson's equation $\vec{\nabla} \cdot \vec{g} = -4\pi G\rho$.
Ψ is determined once $\vec{u}$ and $\overleftrightarrow{\pi}$ are specified.
$\Gamma - \Lambda$ can be computed from properties of the gas and the radiation field.

This leaves us with 5 equations and 14 unknowns: ρ, $\vec{u}$, P, $\overleftrightarrow{\pi}$ (five independent elements), ϵ, $\vec{F}$.
To get a closed set of equations, we must find relations among ρ, P, ϵ, $\overleftrightarrow{\pi}$, and $\vec{F}$ using constitutive relations for the gas.
We have already discussed the simplest example, the equation of state for a monatomic gas, $\epsilon = \frac{3}{2}\frac{P}{\rho}$.
We will return to a more general discussion of the equation of state and to viscosity and heat conduction soon.

7. The Euler equations

Formally, the Navier-Stokes equations can be derived from the first-order expansion of the Boltzmann equation in the parameter λ/L.
In many cases, it is adequate to use the zero'th-order approximation $\lambda/L = 0$, in which case one gets the eight "constraint" equations $\overleftrightarrow{\pi} = 0$, $\vec{F} = 0$.
Viscosity and heat conduction are diffusive effects (arising because particles diffuse out of their fluid elements), and these are often small compared to dynamical effects, when flows are a significant fraction of (or greater than) the sound speed, or when diffusion times are long compared to system lifetimes.
Some important exceptions in astrophysical situations are shocks, viscosity in accretion disks, conductive evaporation of cold clouds in a hot medium, and conduction in some kinds of stars and planets.

If we neglect diffusive terms in the Navier-Stokes equations, we get the Euler equations

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{\partial\rho}{\partial t} + \vec{u}\cdot\vec{\nabla}\rho = -\rho\vec{\nabla}\cdot\vec{u}, \tag{14}$$

$$\frac{\mathrm{D}\vec{u}}{\mathrm{D}t} = \frac{\partial\vec{u}}{\partial t} + \left(\vec{u}\cdot\vec{\nabla}\right)\vec{u} = \vec{g} - \frac{1}{\rho}\vec{\nabla}P \tag{15}$$

$$\frac{\mathrm{D}\epsilon}{\mathrm{D}t} = \frac{\partial\epsilon}{\partial t} + \vec{u}\cdot\vec{\nabla}\epsilon = -\frac{P}{\rho}\vec{\nabla}\cdot\vec{u} + \frac{\Gamma - \Lambda}{\rho} \tag{16}$$

Together with the equation of state $\epsilon = \frac{3}{2}\frac{P}{\rho}$, the Euler equations describe the dynamics of a perfect monatomic gas.
In a perfect gas, collisions ensure that the local distribution of random velocities acquires the maximum entropy form, in which the velocity distribution in each dimension is independently Gaussian with variance $\langle w_i^2\rangle = \frac{kT}{m}$.
This is a Maxwellian velocity distribution,

$$f_{\mathrm{Max}}(\vec{w})d3w = \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{mw2}{2kT}\right) d3w. \tag{17}$$

The Maxwellian distribution has $\langle w_i w_j\rangle = 0$ for $i \neq j$ and $\langle w^2 w_i\rangle = 0$, implying $\overleftrightarrow{\pi} = 0$ and $\vec{F} = 0$.

8. The Jeans Equations

It is worth briefly noting the analogy between the Euler equations of fluid dynamics and the Jeans equations of stellar dynamics. These are also derived by taking moments of the Boltzmann equation, in this case the collisionless Boltzmann equation (Binney & Tremaine, section 4.2.)

The first Jeans equation is just the continuity equation for stellar number density, exactly equivalent to the fluid continuity equation for mass density.

The second Jeans equation, in Binney & Tremaine's notation, is

$$\nu \frac{\partial \overline{v}_j}{\partial t} + \sum_i \nu \overline{v}_i \frac{\partial \overline{v}_j}{\partial x_i} = -\nu \frac{\partial \Phi}{\partial x_j} - \sum_i \frac{\partial (\nu \sigma_{ij}^2)}{\partial x_i}, \tag{18}$$

where $\overline{\mathbf{v}}$ is the mean fluid velocity, ν is the stellar number density, Φ is the gravitational potential, and $\sigma_{ij}^2 = \langle w_i w_j \rangle$ where $\vec{w}$ is the random velocity.
This is very similar to the second Euler equation, but with an anisotropic stress tensor $\nu \sigma_{ij}^2$ in place of $\vec{\nabla} P$.

The limitation of the Jeans equations is that there is no equation of state relation between density and σ_{ij}^2, no equivalent of the thermal energy equation, and no strong reason for assuming a Maxwellian random velocity distribution.
Thus, stellar dynamical models that use the Jeans equations must usually assume some form of σ_{ij}^2, and the resulting models are only as accurate as the assumptions.